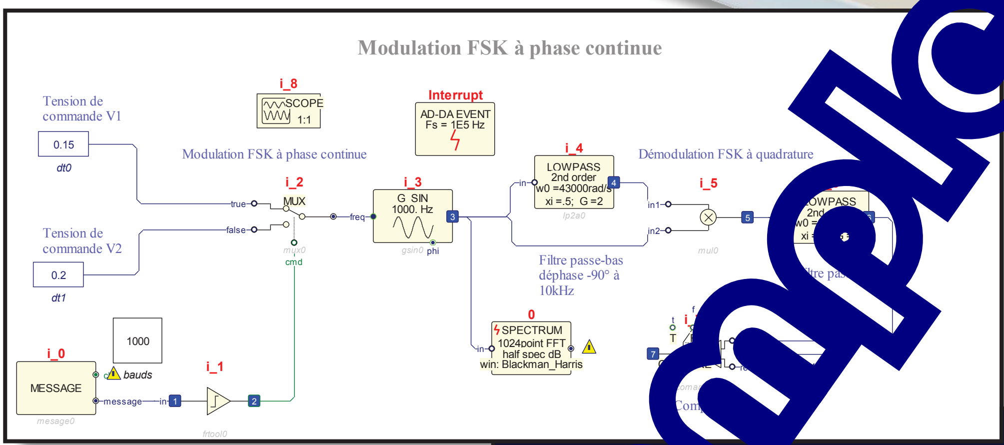
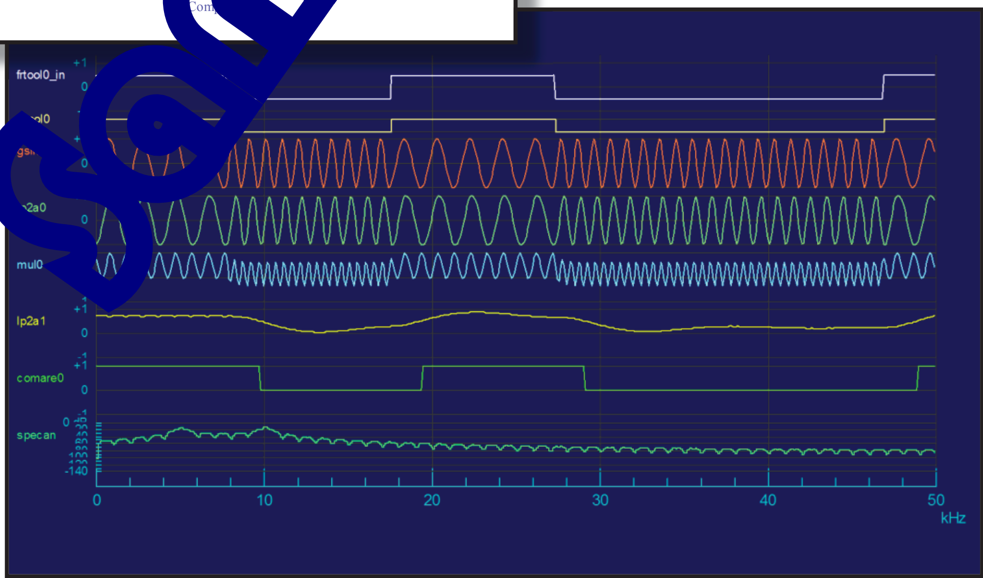
Signal Processing

Level IV/V CITE, BTS/DUT/Licence.





Topics and Reports

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SUMMARY

Ex.4 :
SAMPLE: NYQUIST -SHANON THEOREM

4.1 TARGET

At usually regular instants, pick up the values taken by a signal for various types of treatment; examples:

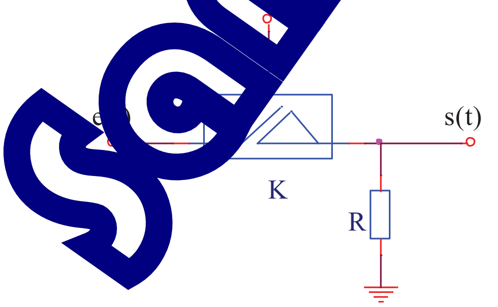
1- Wired or radio transmission, time-division multiplexing, and analog processing,
2- Digital processing (optionally followed by a restitution of the analog signal) needs to constantly maintain the value of the samples during their conversion in digital information; this requires functions :
   - Sample/Hold,
   - Analog-digital conversion ADC,
   - Digital processing : computer or DSP
   - Digital- analog conversion if necessary.

In any case, it must be ensured that the processed s[illegible] substantially altered by sampling operation:

- limited information loss
- no change in the spectrum , that is [illegible] no [illegible]dded information

4.2 SAMPLE PRINCIPLE DIAGRAM

The diagram of the sampling fun[illegible] igure 1.



e(t) : input signal.
h(t) : sampling signal. The rectangular signal h (t) is defined as:

$$h(t) = H \; for 0 < t \leq \tau$$
$$h(t) = 0 \quad for \tau < t < Ts$$

s(t) : output signal
K : switch controlled by h (t) assumed to be perfect.

4.3 FUNCTION

The h(t) signal has 2 states : a high state and a low state.

a) h(t) is in the state (for example high state), K is closed so s(t) = e(t)

b) h(t) is in another state (low state), K is opened, s(t) = 0.
Consider that h(t) as an amplitude 1 or 0 signal.
The operation may be assimilated to **a multiplication operation see EX modulation:**

$$\boldsymbol{s(t) = k.h(t).e(t)}$$

K Unit: $[V]^{-1}$

If the amplitude of h(t) is 1, then the module k will be 1.
The expression for s (t) is determined from the decomposition in h (t) Fourier series.

If h(t) is a periodic signal of T_e or T_s period (figure 2), it may be decomposed in Fourier series.

$$T_s = \frac{1}{F_s} = \frac{2\pi}{\omega_s}$$

$$h(t) = \sum_{n=0}^{n=\infty} [a_n \cos(n\omega_s t) + b_n \sin(n\omega_s t)]$$

The signal h (t) can be formed as:

$$h(t) = \sum_{n=0}^{n=\infty} A_n \cos(n\omega_s t + \varphi_n) \quad equation\ (a)$$

with

$$A_n = \sqrt{a_n^2 + b_n^2} \quad and \quad \varphi_n = Arctan(\frac{b_n}{a_n})$$

$$A_0 = \alpha$$

QUESTION :

Determine the coefficients expression of Fourier series;

ANSWER

For n ranging from 0 to infinity, the coefficients of the series are calculated as follows:

$$\boldsymbol{a_0 = \frac{1}{T_s}\int_0^{T_s} h(t)dt = H\frac{\tau}{T_s} = \alpha}$$

$$\alpha\ is\ the\ cycle\ duty\ of\ h(t)$$

$$a_n = \frac{2}{T_s}\int_0^{T_s} h(t)\cos(n\omega_s t)\,dt \quad b_n = \frac{2}{T_s}\int_0^{T_s} h(t)\sin(n\omega_s t)\,dt$$

$$a_n = \frac{2}{T_s}\int_0^{\tau} \cos(n\omega_s t)\,dt = \frac{2}{n\omega_s T_s}[sin(n\omega_s t)]_0^{\tau}$$

$$b_n = \frac{2}{T_s}\int_0^{\tau} \sin(n\omega_s t)\,dt = \frac{2}{n\omega_s T_s}[-cos(n\omega_s t)]_0^{\tau}$$

$$a_n = \frac{2}{nT_s\omega_s} sin(n\omega_s \tau)$$
$$b_n = \frac{2}{nT_s\omega_s}(1 - cos(n\omega_s \tau))$$

$$T_s\omega_s = 2\pi$$

This gives for the coefficients, the following expressions:

$$a_n = \frac{1}{n\pi}\sin(2n\pi\frac{\tau}{T_s}) = \frac{1}{n\pi}\sin(2n\pi\alpha)$$

$$b_n = \frac{1}{n\pi}(1 - \cos(2n\pi\frac{\tau}{T_s})) = \frac{1}{n\pi}(1 - \cos(2n\pi\alpha))$$

A_n is the amplitude of the spectral line of h(t).

a) If α is equal to 0.5, the coefficients a_n are null.
Only the b_n coefficients remain: the fundamental F_s and the impaired harmonics of the form $nF_s = (2p+1)F_s$ whose amplitude A_{2p+1} is given by the relation :

$$\boldsymbol{b_n = A_{2p+1} = \frac{2}{(2p+1)\pi}}$$

b) If α is any fundamental, and all harmonics are present and are for the amplitude, the An is :

$$A_n^2 = a_n^2 + b_n^2$$

$$A_n^2 = \frac{1}{(n\pi)^2}(\sin^2(2n\pi\alpha) + (1 - \cos(2n\pi\alpha)^2)$$
$$A_n^2 = \frac{1}{(n\pi)^2}(\sin^2(2n\pi\alpha) + \cos^2(2n\pi\alpha) - 2\cos(2n\pi\alpha))$$

This gives:

$$A_n^2 = \frac{2}{(n\pi)^2}(1 - \cos(2n\pi\alpha))$$
$$with$$
$$1 - \cos(2n\pi\alpha) = 2sin^2(n\alpha\pi)$$

$$Then$$

$$A_n^2 = \frac{2}{(n\pi)^2} 2sin^2(n\alpha\pi) = \frac{4}{(n\pi)^2} sin^2(n\alpha\pi)$$

$$A_n^2 = \frac{4}{(n\pi)^2} sin^2(n\alpha\pi)$$

$$A_n = \frac{2}{n\pi} \sin(n\alpha\pi)$$

By multiplying by $\frac{\alpha}{\alpha}$, we obtain ;

$$\boldsymbol{A_n = 2\alpha \frac{\sin(n\alpha\pi)}{n\alpha\pi}}$$

Example : amplitude h(t) = 1V.

A_n Coefficients in terms of α for n = 1

α	0,1	0,2	0,3	0,4	0,[illegible]	[illegible]	0,7	0,8	0,9
$2\alpha \frac{sin(\alpha\pi)}{\alpha\pi}$	0,1967	0,3742	0,5150	0,60[illegible]	0,[illegible]6	[illegible]6055	0,5150	0,3742	0,1967

A_n Coefficients in terms of n for α = 1

n	1	2	3	4	5	6	7	8	9
$2\alpha \frac{sin(n\alpha\pi)}{n\alpha\pi}$	0,6055	0,18[illegible]	0,12[illegible]	[illegible]14	0,0000	0,1009	0,0535	0,0468	0,0673

Remark:

The amplitude of the lines is zero for nα = k integer: sin (nαπ) = sin (kπ) = 0

4.4 EXPRESSION OF SAMPLED SIGNAL s(t)

To simplify the calculation, we suppose that e(t) is a pure sinusoidal signal with the pulsation ω (and the frequency f = ω/2π).

From the superposition theorem, the calculation can be applied to any signal.

$$e(t) = E_m sin\omega t = E_m sin2\pi f t$$

$$s(t) = e(t).h(t)$$

QUESTION

1- *Determine the expression of the sampled signal*
2- *Show that the spectrum is composed of the frequency f and an infinite number of frequency lines* $nf_s \pm f$ *for n ranging from 1 to infinity.*

ANSWER

1- With : $\omega_s = 2\pi f_s$

$$s(t) = \sum_{n=0}^{n=\infty} (E_m sin\omega t) A_n \cos(n\omega_s t - \varphi_n)$$

$$s(t) = E_m \sum_{n=0}^{n=\infty} A_n \, sin\omega t . \cos(n\omega_s t - \varphi_n)$$

$$s(t) = \alpha E_m sin\omega t + \sum_{n=1}^{n=\infty} A_n sin\omega t . \cos(n\omega_s t - \varphi_n)$$

By $\omega = 2\pi f$ and $\omega_s = 2\pi f_s$, we obtain:

$$s(t) = \alpha E_m sin2\pi ft + \frac{1}{2}\sum_{n=1}^{n=\infty} A_n[\sin((n2\pi fs + 2\pi f)t - \varphi_n) + \sin((n2\pi fs - 2\pi f)t - \varphi_n)]$$

$$\boldsymbol{s(t) = \alpha E_m sin2\pi ft + \frac{1}{2}\sum_{n=1}^{n=\infty} A_n[\sin(2\pi(nfs + f)t - \varphi_n) + \sin(2\pi(nfs - f)t - \varphi_n)]}$$

2- We note that the spectrum is composed:
   1- Of the frequency **f**
   2- Of the frequencies **nfs–f and nfs + f for n ranging from 1 to infinity.**

4.5 COMPLEX WRITING OF FOURIER SERIES: BI-LATERAL SERIES

The Fourier series term of a periodic function f(t), (see the previous), of T period and of ω pulsation is :

$$f(t) = a_0 + \sum_{n=1}^{n=\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

Let: $$u_n = a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

By expressing $\cos(n\omega t) + \sin(n\omega t)$ in their complex form, we obtain:

$$u_n = a_n \frac{e^{jn\omega t} + e^{-jn\omega t}}{2} + b_n \frac{e^{jn\omega t} - e^{-jn\omega t}}{2j}$$

QUESTION

1- *Determine the expression of the complex coefficients of the Fourier series:* *C_n and $C_{-n} = \overline{C_n}$ (conjugate complex of C_n).*
2- *Find the modules of C_n and C_{-n} in terms of A_n coefficients of the unilateral series.*

ANSWER

1- By multiplying the numerator and denominator with the term of the second member by –j and putting the exponential as factor, we obtain :

$$u_n = \frac{a_n - jb_n}{2} e^{jn\omega t} + \frac{a_n + jb_n}{2} e^{-jn\omega t}$$

Let:

$$C_n = \frac{a_n - jb_n}{2} \text{ and } C_{-n} = \frac{a_n + jb_n}{2} \text{ with:}$$

$$C_{-n} = \overline{C_n} \text{ two complex conjugate numbers .}$$

So it shows the relation of u_n:

$$\boldsymbol{u_n = C_n e^{jn\omega t} + C_{-n} e^{-jn\omega t}}$$

This gives the relation for f(t):

$$f(t) = a_0 + \sum_{n=1}^{n=\infty} u_n = a_0 + \sum_{n=1}^{n=\infty} [C_n e^{jn\omega t} + C_{-n} e^{-jn\omega t}]$$

This expression can be written in simpler form <u>**by make the n range from -∞ to +∞**</u>:

$$\boldsymbol{f(t) = \sum_{n=-\infty}^{n=\infty} C_n e^{jn\omega t}}$$

Retake the previous Fourier series coefficients calculation, but this time the series is under the exponential form. We obtain:

$$a_n = \frac{2}{T}\int_a^{a+T} \mathrm{f(t)cos}(n\omega t)\, dt = \frac{2}{T}\int_a^{a+T} f(t)\frac{e^{n\omega t} + e^{-n\omega t}}{2} dt (1)$$

$$\boldsymbol{a_n} = \frac{2}{T}\int_a^{a+T} f(t)\frac{e^{n\omega t} + e^{-n\omega t}}{2} dt = \boldsymbol{\frac{1}{T}\int_a^{a+T} f(t)(e^{n\omega t} + e^{-n\omega t}) dt (2)}$$

$$b_n = \frac{2}{T}\int_a^{a+T} \mathrm{f(t)sin}(n\omega t)\, dt = \frac{2}{T}\int_a^{a+T} f(t)\frac{e^{n\omega t} - e^{-n\omega t}}{2j} dt \quad (3)$$

$$jb_n = \frac{2j}{T}\int_a^{a+T} f(t)\frac{e^{n\omega t} - e^{-n\omega t}}{2j}dt = \frac{1}{T}\int_a^{a+T} f(t)(e^{n\omega t} - e^{-n\omega t})dt(4)$$

The operation $\frac{(3)-(4)}{2} = \frac{an-jbn}{2}$ leads:

$$C_n = \frac{a_n - jb_n}{2} = \frac{1}{2}\left(\frac{1}{T}\int_a^{a+T} 2f(t)e^{-n\omega t}dt\right)$$

$$C_n = \frac{1}{T}\int_a^{a+T} f(t)e^{-n\omega t}dt$$

In the same way, the operation $\frac{(3)+(4)}{2} = \frac{an+jbn}{2}$ results in:

$$C_{-n} = \frac{1}{T}\int_a^{a+T} f(t)e^{n\omega t}dt$$

In summary, we will have:

$$f(t) = \sum_{n=-\infty}^{n=+\infty} C_n e^{jn\omega t}$$

$$C_n = \frac{a_n - jb_n}{2} = \frac{1}{T}\int_a^{a+T} f(t)e^{-n\omega t}dt$$

$$C_{-n} = \frac{a_n + jb_n}{2} = \frac{1}{T}\int_a^{a+T} f(t)e^{+n\omega t}dt$$

$$C_0 = \frac{1}{T}\int_a^{a+T} f(t)dt$$

2- Calculate the modules of bilateral spectrum elements of the f(t) function.

$$|C_n|^2 = \frac{a_n^2 + b_n^2}{4} = A_n^2$$

$$|C_n| = \frac{A_n}{2}$$

Important remark

It is found that the amplitude of the bilateral spectrum components is half of that of unilateral spectrum.

4.7 PRACTICAL WORKS

4.7.1 Manipulation Purpose:

The purpose of the manipulation is to demonstrate the Nyquist-Shannon theorem.

QUESTION

State the Nyquist-Shannon theorem.

ANSWER

Nyquist-Shannon theorem

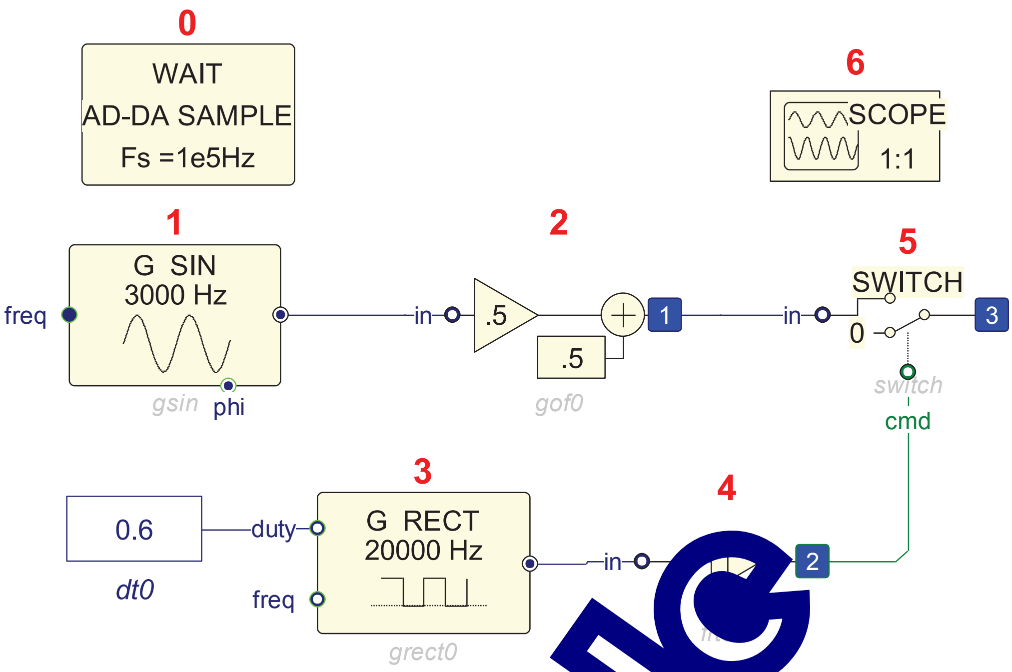
We don't lose the information by reconstructing a signal from its samples if the sampling frequency is at least equal to twice the highest contained frequency in the spectrum of the sampled signal.

4.7.2 Application diagram

Realize the application diagram as figure 2.
The figure 2 indicates the information defining the sampling parameters.
The figures 3 to 9 can demonstrate the effect of calculations in the theoretical part.

4.7.3 Nyquist-Shannon theorem demonstration

The figures 12 to 15 can demonstrate the recovery or folding of sampled signal thanks to the module edt1.
The figure 16.allowed to make the message spectrum vary by acting on the module dt1 by means of the mouse; observe at the same time the spectrum cover on the figure 16.b.



Nyquist_shannon_

fig.

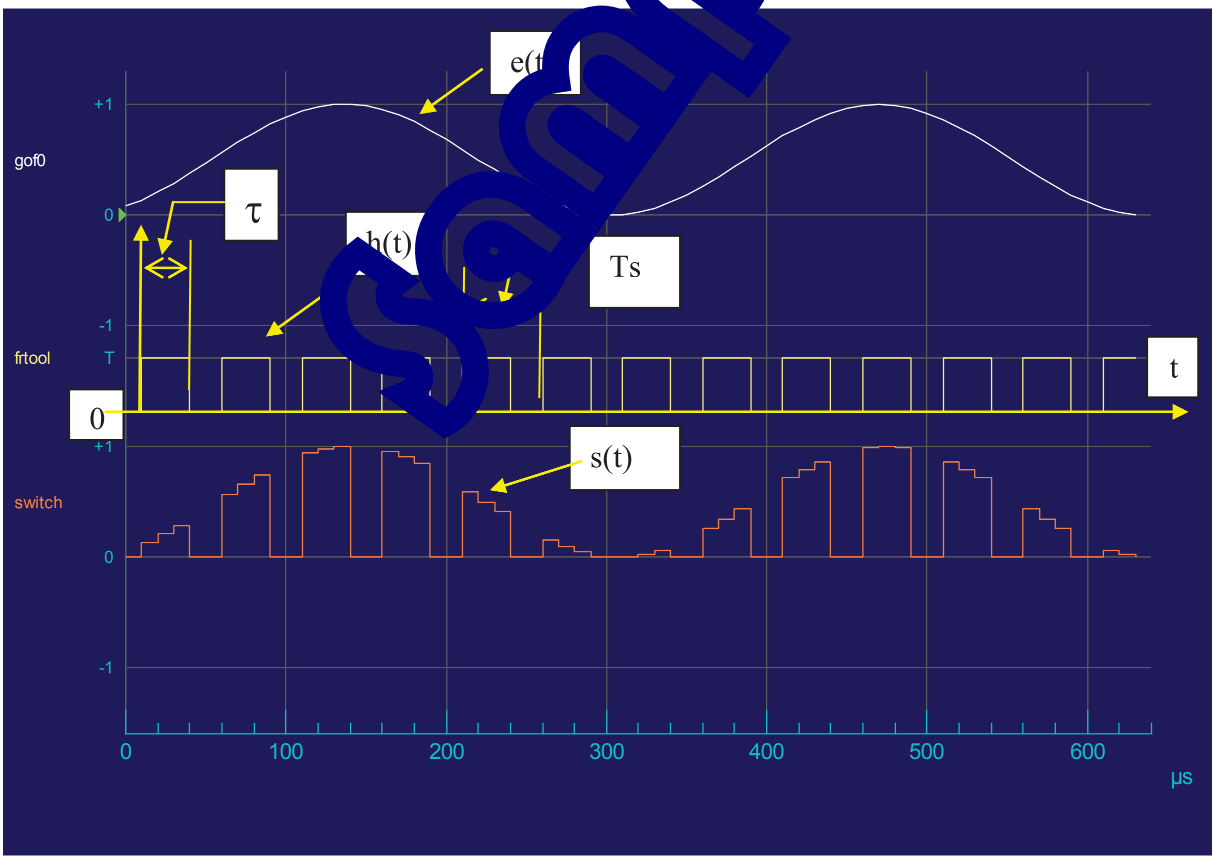
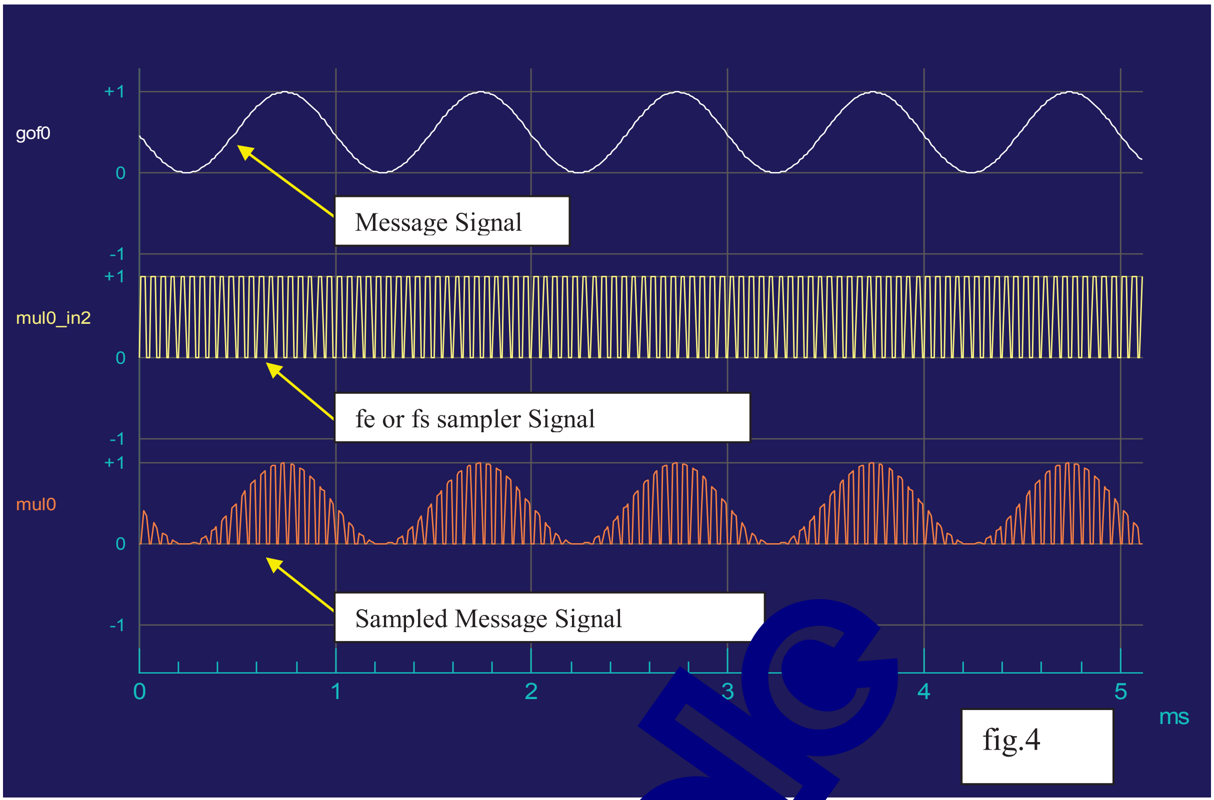
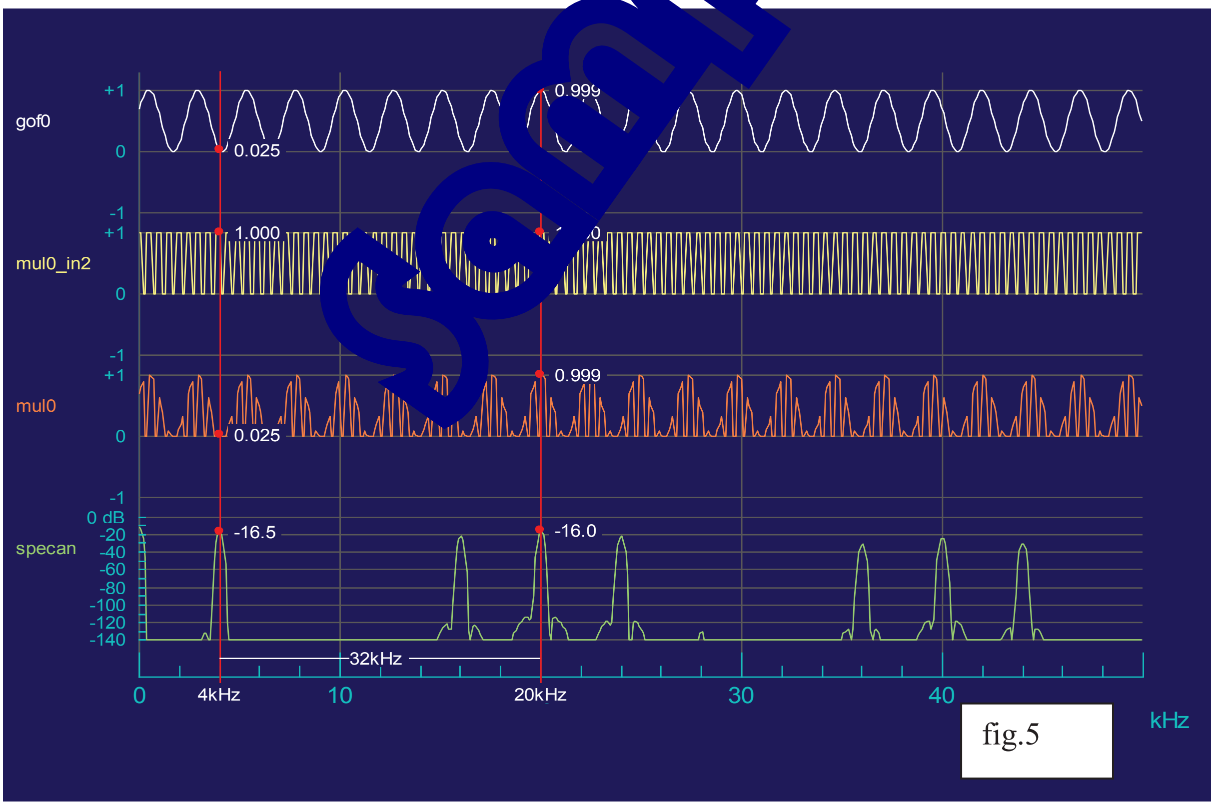


fig.3



Message Signal: [illegible] wave



Spectrum of the sampled message: fs = 20kHz fm = 4kHz

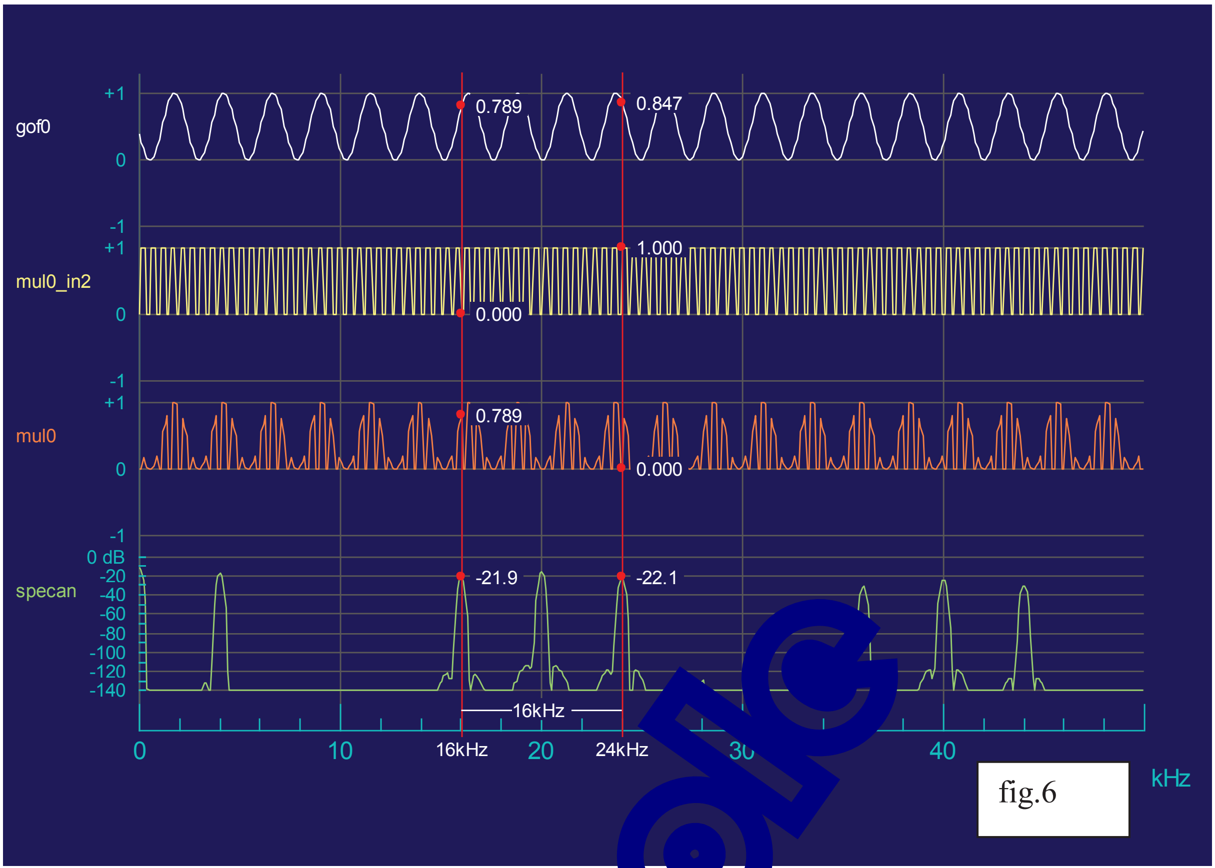


fig.6

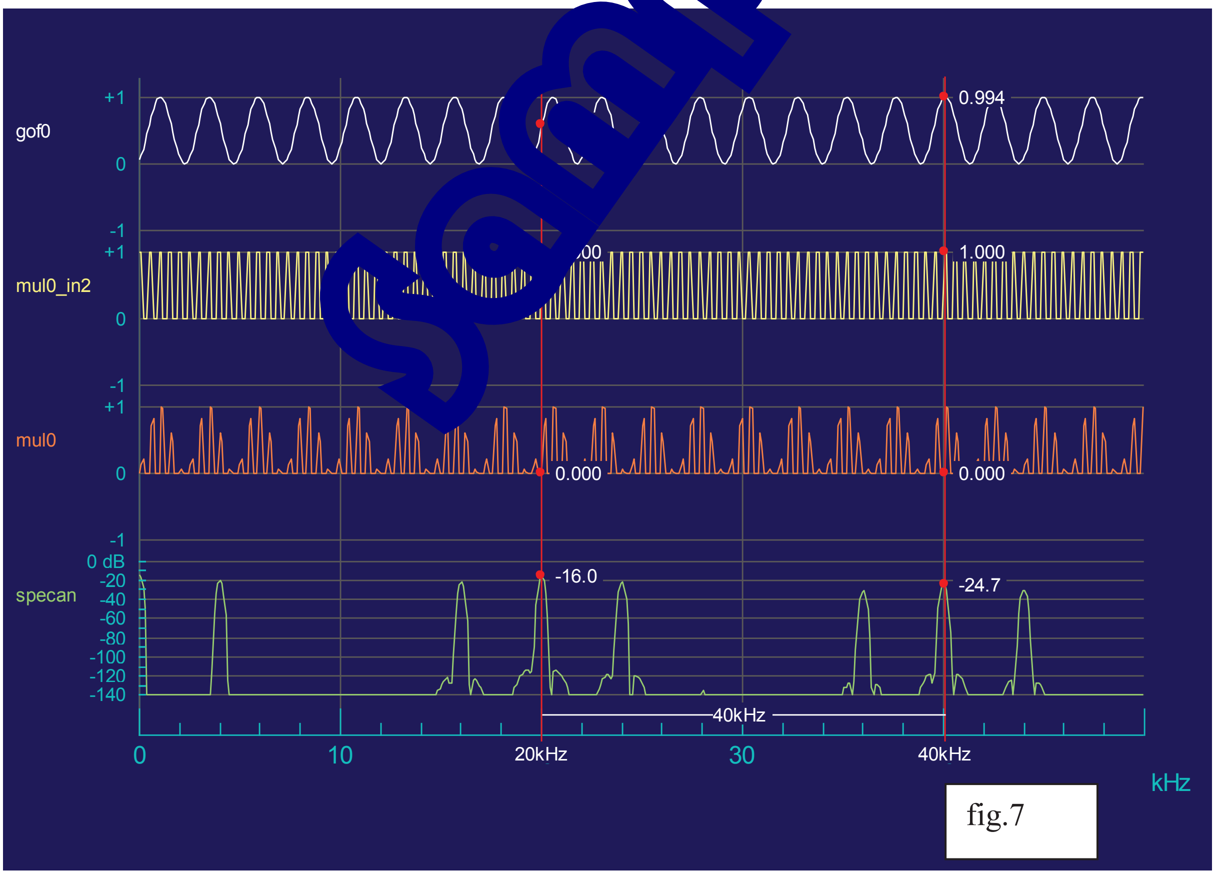


fig.7

Spectrum : fm,   fs-fm, n.fs, fs+fm, repeated every fs : 20kHz, 40 kkHz

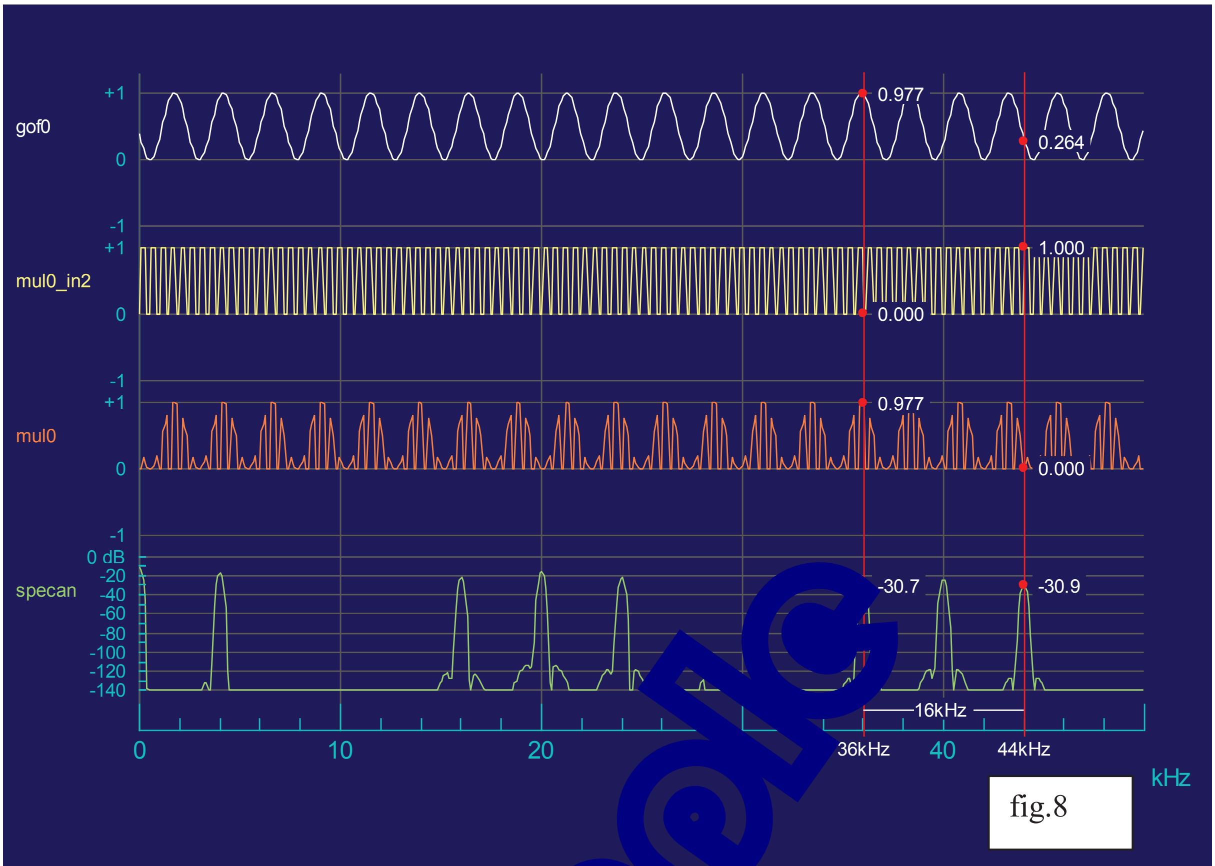

gof0
+1
0
-1
0.977
0.264
mul0_in2
1.000
0.000
mul0
0.977
0.000
specan
0 dB
-20
-40
-60
-80
-100
-120
-140
-30.7
-30.9
16kHz
0
10
20
36kHz
40
44kHz
kHz

fig.8

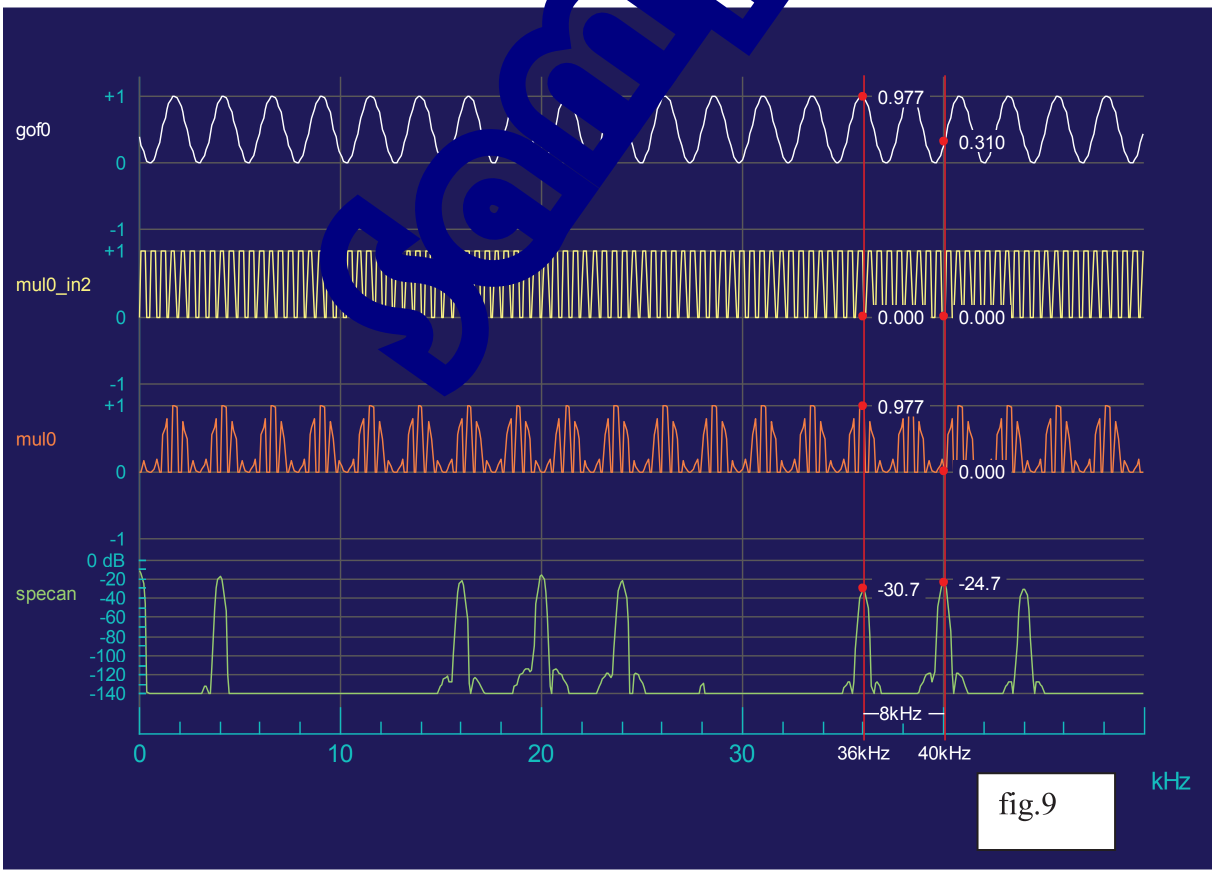
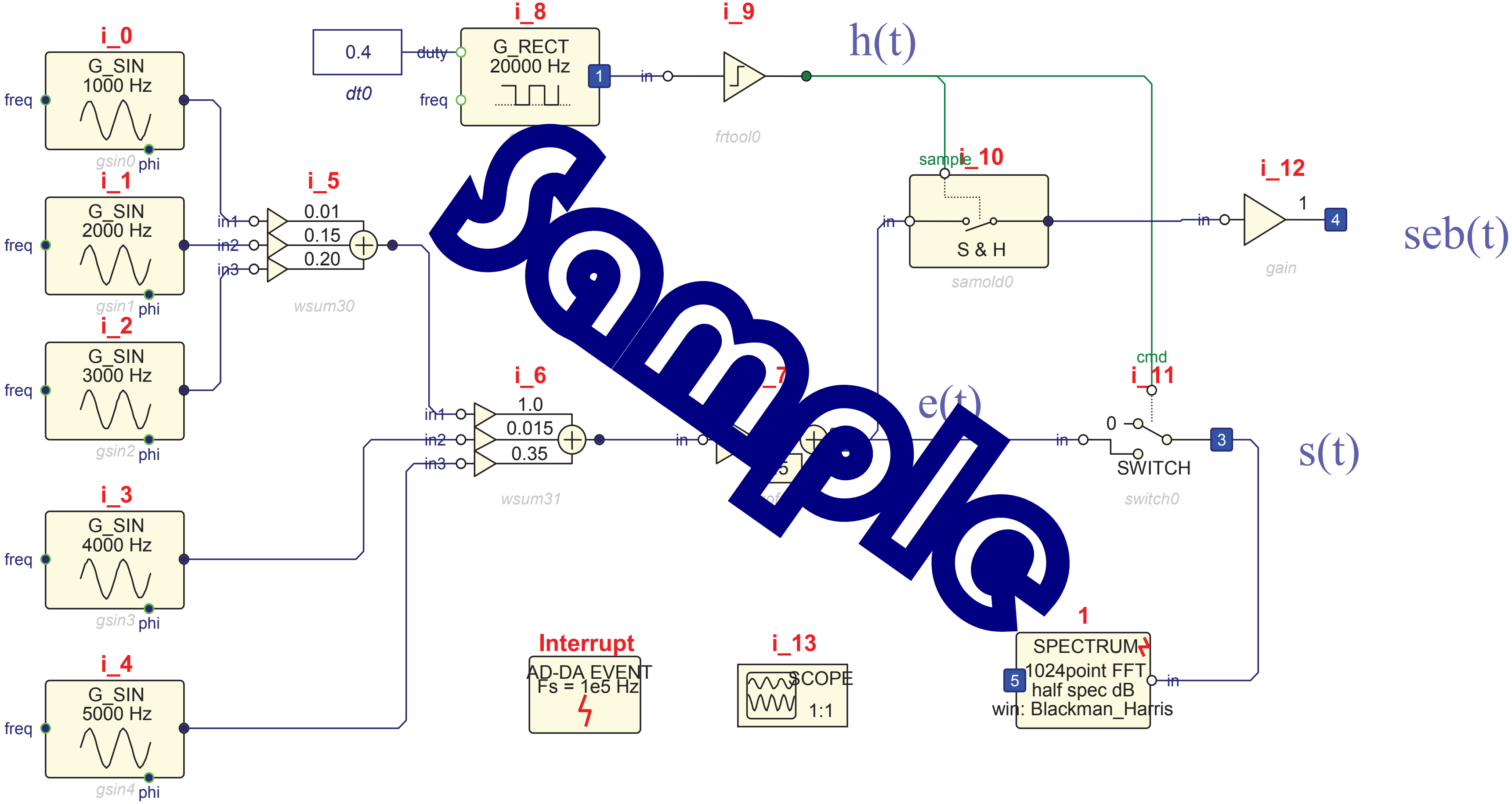

gof0
+1
0
-1
0.977
0.310
mul0_in2
0.000
0.000
mul0
0.977
0.000
specan
0 dB
-20
-40
-60
-80
-100
-120
-140
-30.7
-24.7
8kHz
0
10
20
30
36kHz
40kHz
kHz

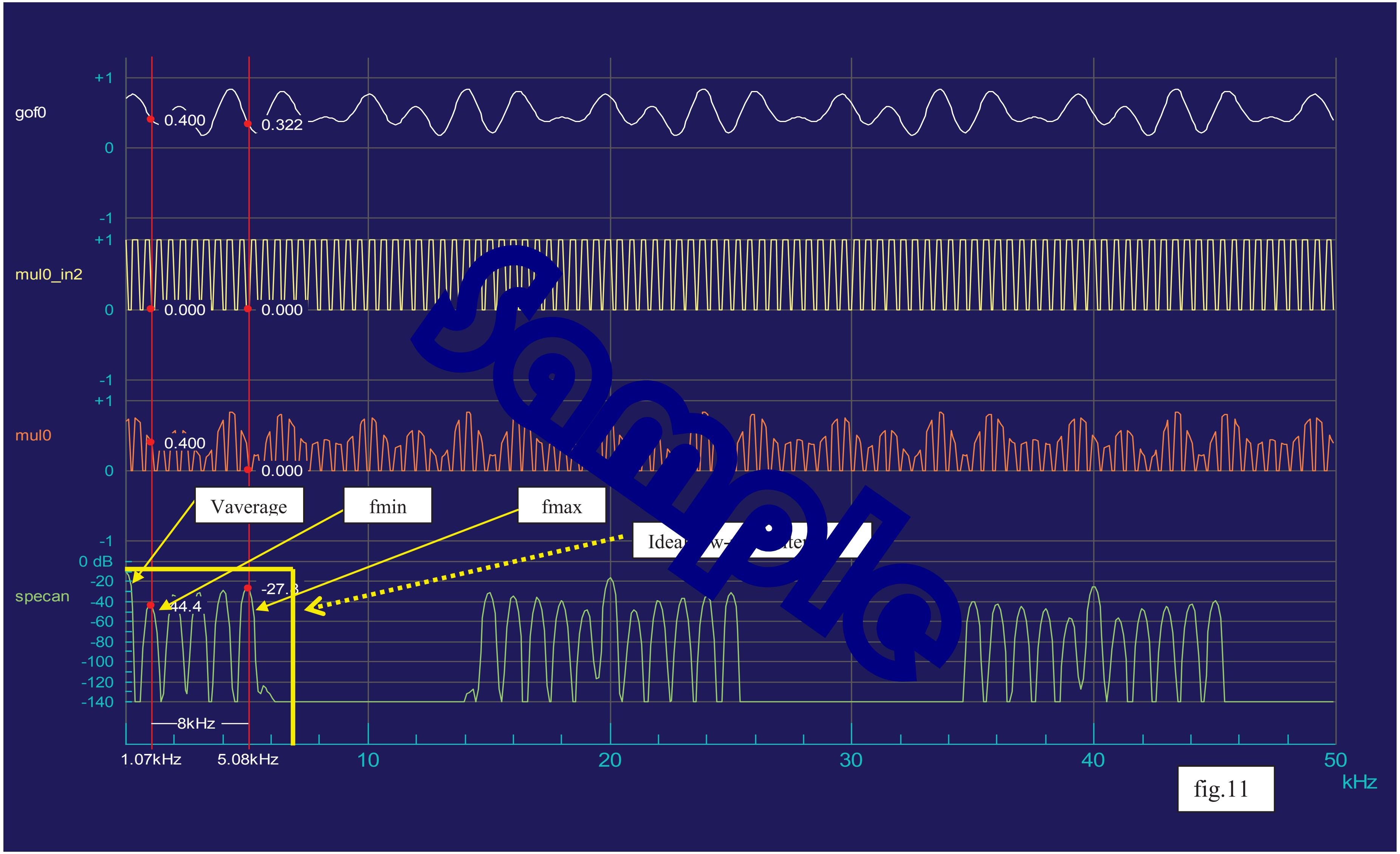
fig.9

Spectrum of a sampled or sampled/blocked signal



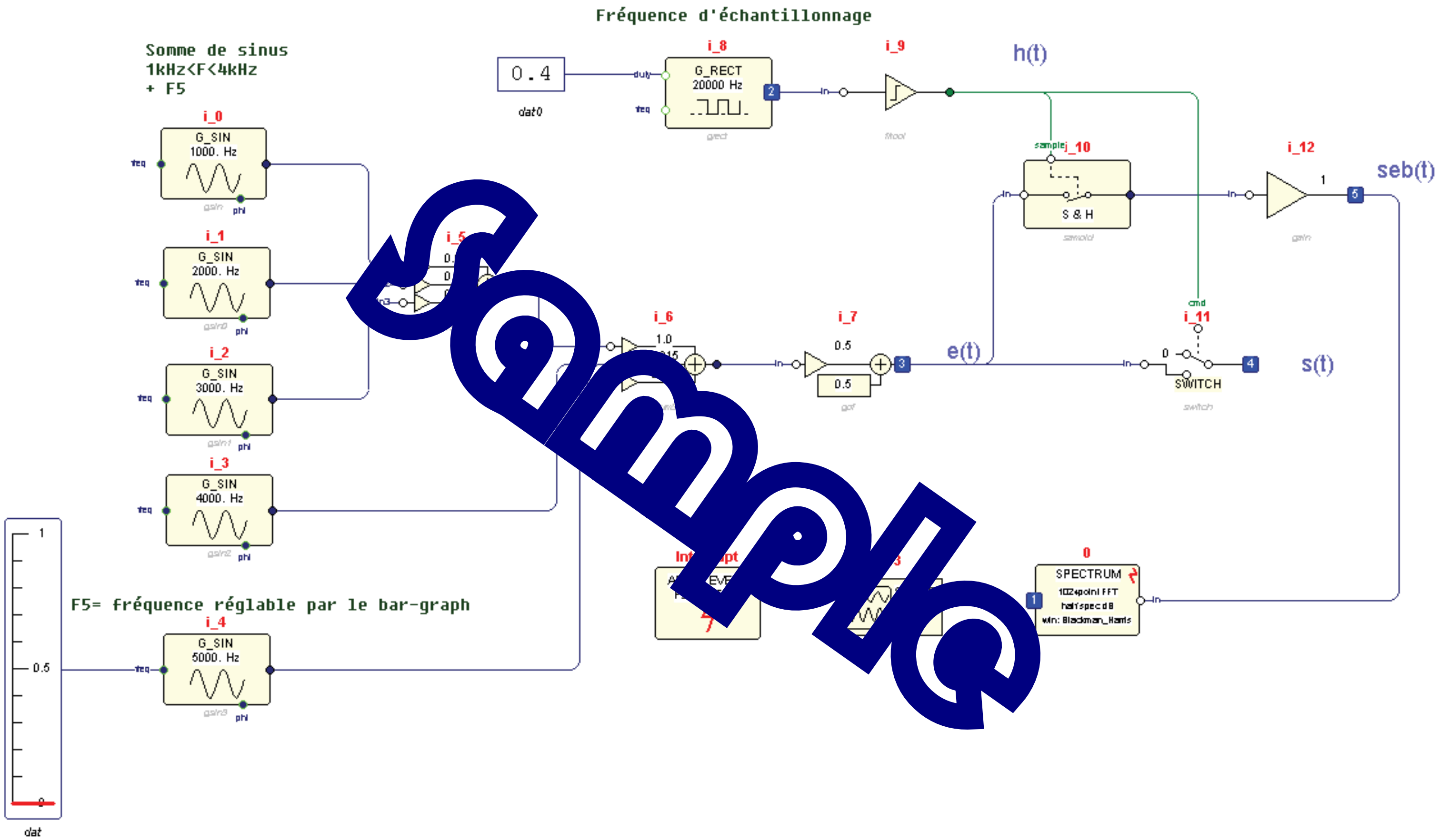
Occupation_Spectrale.fib

fig.10



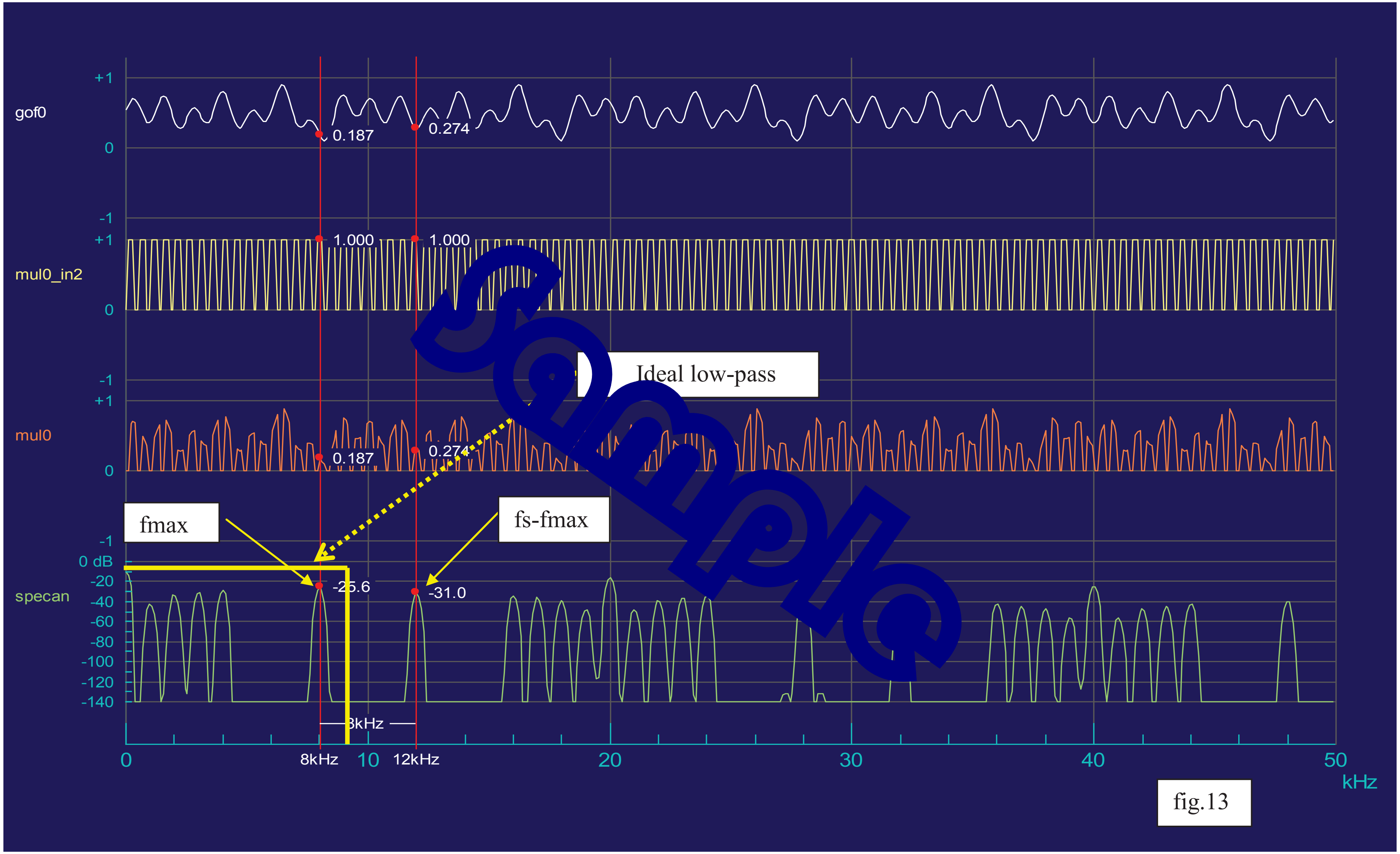
Spectrum of the sampled message signal : fmin = 1kHz ; fmax = 5kHz ; aliasing but no spectrum recovery

Spectre d'un signal variable échantillonné:
Mise en evidence du repliement spectral

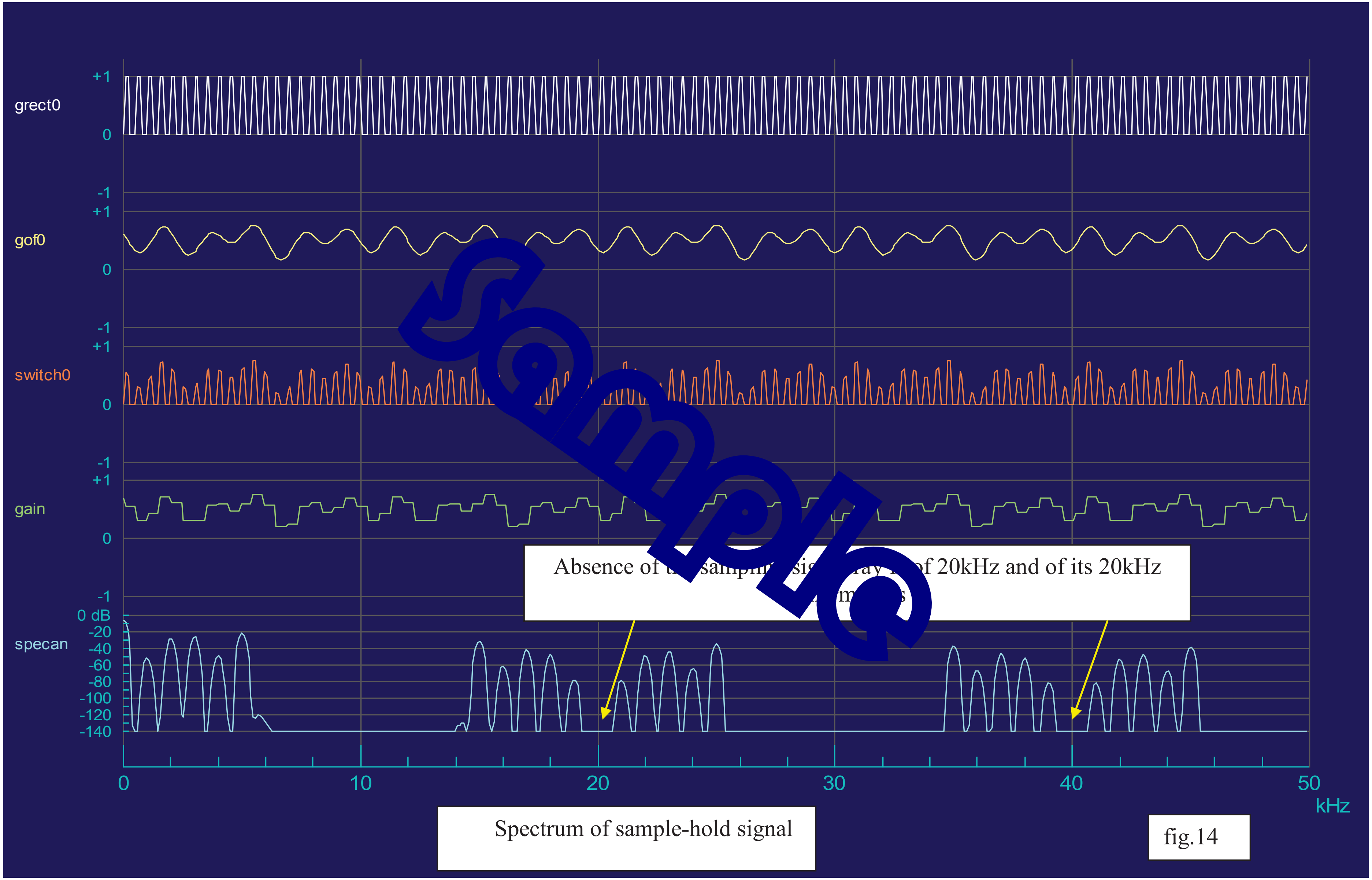


Spectral_recovery.fib

fig.12



fmax =8kHz ➔ fmax < fp-fmax ➔ fmax < fp-fmax ; aliasing but no spectrum overlap. Recoverable Message



Spectrum of sample-hold signal

fig.14

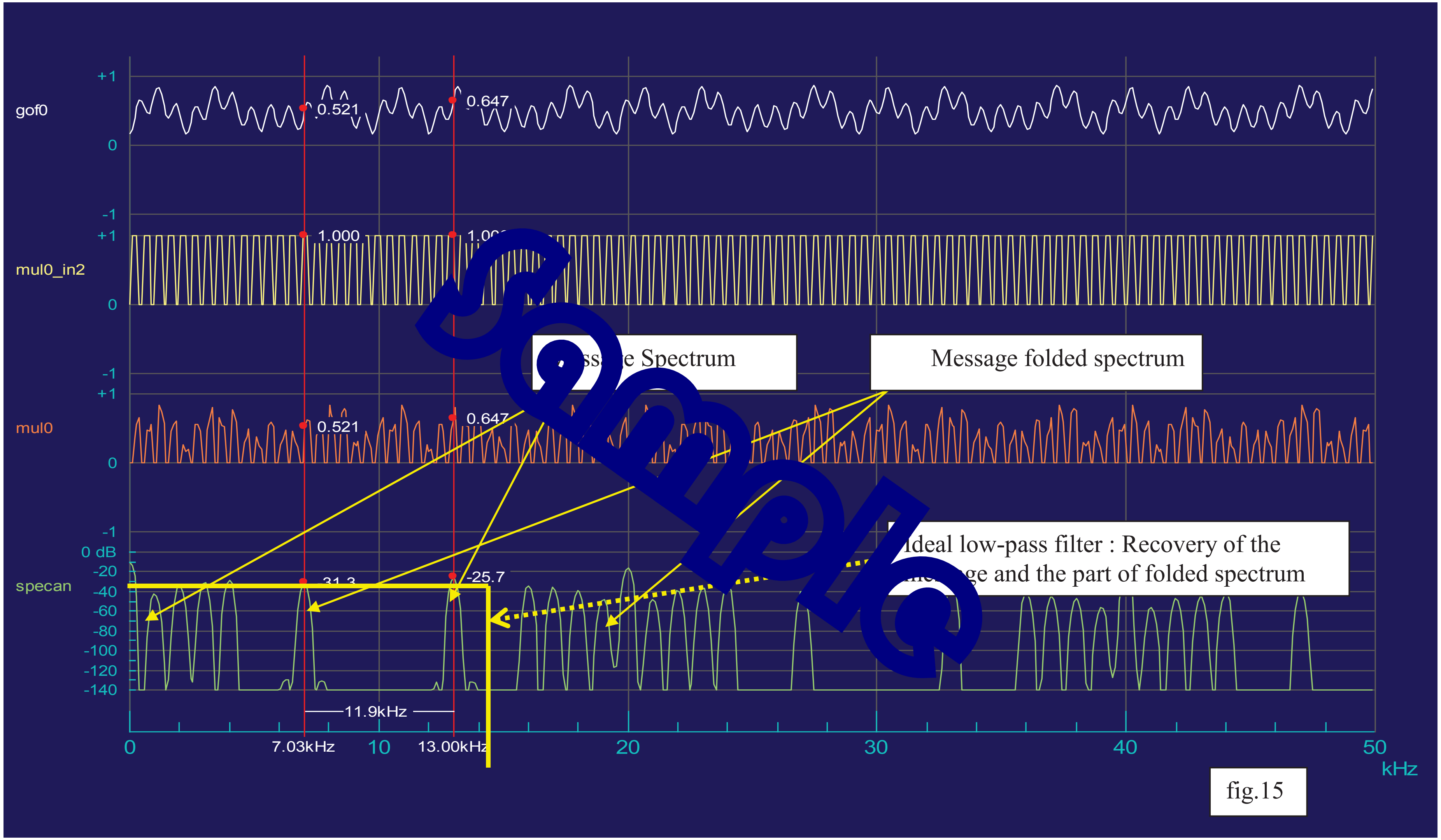


fig.15

fmax = 13Khz ➔ fmax > fs-fmax = 7kHz➔ therefore spectrum recovery; the low-pass filter also recovers a part of the folded spectrum

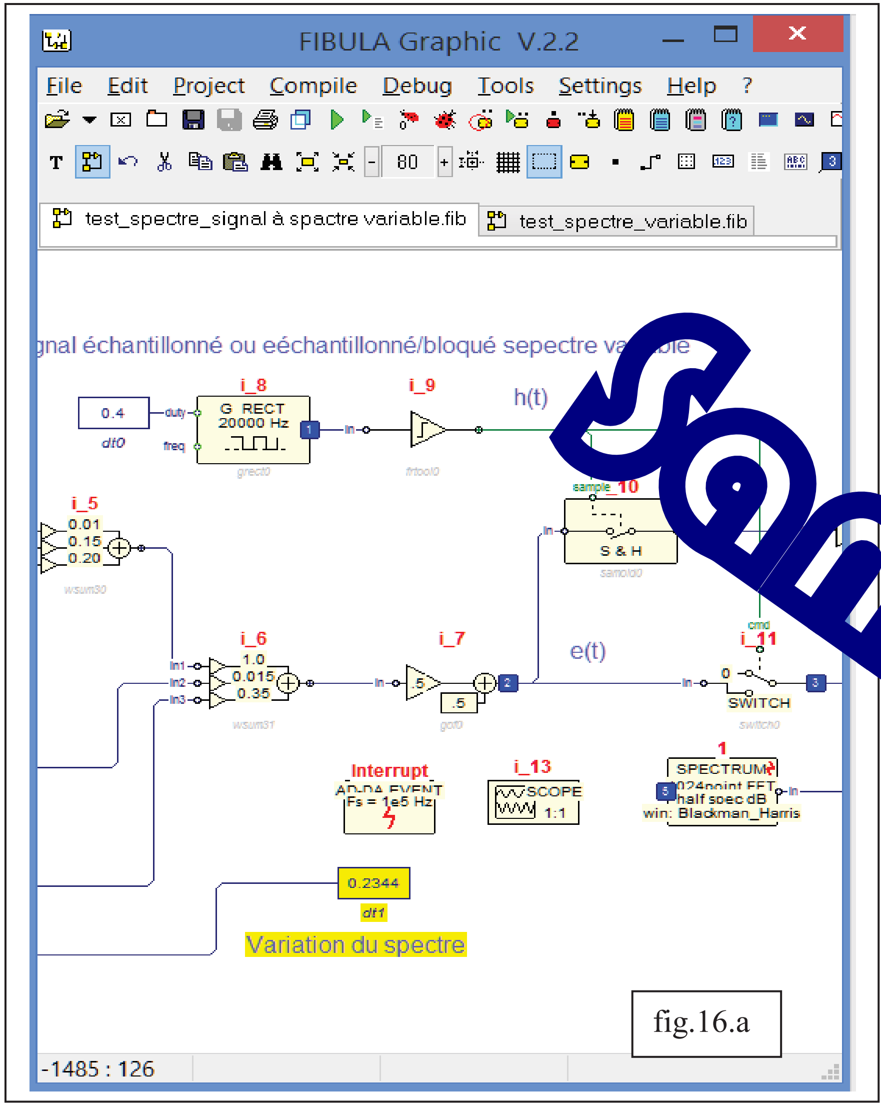


fig.16.a

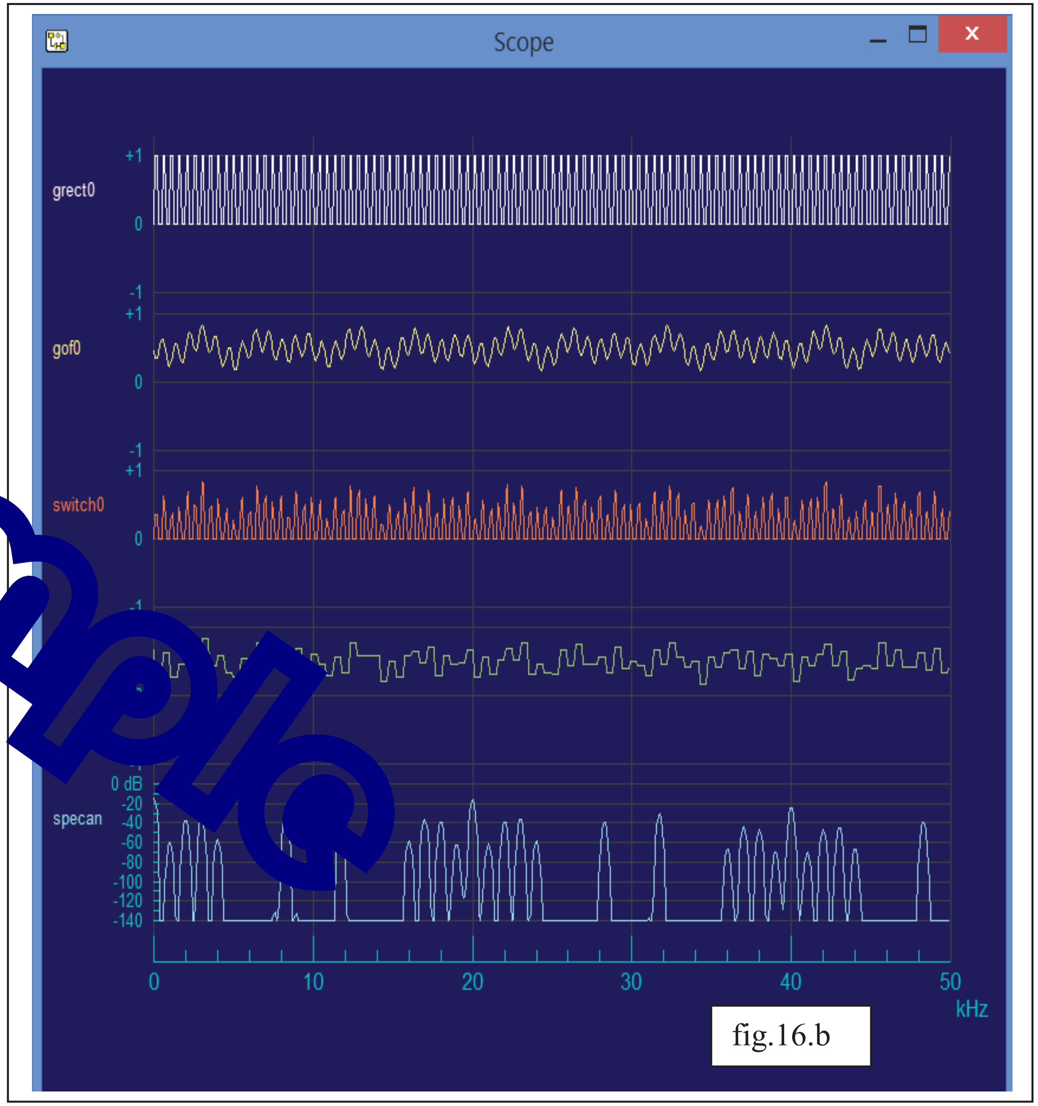


fig.16.b

fig.16